SNSB
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Ergodic Theory and Additive Combinatorics
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Seminar 2

(S2.1) Let (X, T) be a TDS.

(i) Any strongly T-invariant set is also T-invariant.

(ii) The complement of a strongly T-invariant set is strongly T-invariant.

(iii) The closure of a T-invariant set is also T-invariant.

(iv) The union of any family of (strongly) T-invariant sets is (strongly) T-invariant.

(v) The intersection of any family of (strongly) T-invariant sets is (strongly) T-invariant.

(vi) If A is T-invariant, then $T^n(A) \subseteq A$ and $T^n(A)$ is T-invariant for all $n \geq 0$.

(vii) If A is strongly T-invariant, then $T^n(A) \subseteq A$ and $T^{-n}(A) = A$ for all $n \geq 0$; in particular, $T^{-n}(A)$ is strongly T-invariantfor all $n \geq 0$.

(viii) For any $x \in X$, the forward orbit $\mathcal{O}_+(x)$ of x is the smallest T-invariant set containing x and $\overline{\mathcal{O}}_+(x)$ is the smallest T-invariant closed set containing x.

(S2.2) Let (X, T) be an invertible TDS.

(i) $A \subseteq X$ is strongly T-invariant if and only if $T(A) = A$ if and only if A is strongly T^{-1}-invariant.

(ii) The closure of a strongly T-invariant set is also strongly T-invariant.

(iii) If $A \subseteq X$ is strongly T-invariant, then $T^n(A) = A$ for all $n \in \mathbb{Z}$; in particular, $T^n(A)$ is strongly T-invariantfor all $n \in \mathbb{Z}$.

(iv) For any $x \in X$, the orbit $\mathcal{O}(x)$ of x is the smallest strongly T-invariant set containing x and $\overline{\mathcal{O}}(x)$ is the smallest strongly T-invariant closed set containing x.

(v) For any nonempty open set U of X, $\bigcup_{n\in\mathbb{Z}} T^n(U)$ is a nonempty open strongly T-invariant set and $X \setminus \bigcup_{n\in\mathbb{Z}} T^n(U)$ is a closed strongly T-invariant proper subset of X.

(S2.3) Let (X, T) be a TDS and $x \in X$. Then

(i) x is a forward transitive point if and only if $x \in \bigcup_{n\geq 0} T^{-n}(U)$ for every nonempty open subset U of X.

(ii) Assume that (X, T) is invertible. Then x is a transitive point if and only if $x \in \bigcup_{n\in\mathbb{Z}} T^n(U)$ for every nonempty open subset U of X.

(S2.4) Let (X, T) be a TDS with X metrizable and $(U_n)_{n\geq 1}$ be a countable basis of X. Then

(i) $\{x \in X \mid \overline{\mathcal{O}}_+(x) = X\} = \bigcap_{n\geq 1} \bigcup_{k\geq 0} T^{-k}(U_n)$.

(ii) If (X, T) is invertible, then $\{x \in X \mid \overline{\mathcal{O}}(x) = X\} = \bigcap_{n\geq 1} \bigcup_{k\in\mathbb{Z}} T^{k}(U_n)$.

(S2.5) Let (X, T) be a TDS. The following are equivalent:

(i) If U is a nonempty open subset of X such that $T(U) = U$, then U is dense.

(ii) If $E \neq X$ is a proper closed subset of X such that $T(E) = E$, then E is nowhere dense.